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On two-point boundary correlations in the six-vertex model with DWBC

F. Colomo and A. G. Pronko[1]

I.N.F.N., Sezione di Firenze, and Dipartimento di Fisica, Università di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy

Abstract

The six-vertex model with domain wall boundary conditions (DWBC) on an $N \times N$ square lattice is considered. The two-point correlation function describing the probability of having two vertices in a given state at opposite (top and bottom) boundaries of the lattice is calculated. It is shown that this two-point boundary correlator is expressible in a very simple way in terms of the one-point boundary correlators of the model on $N \times N$ and $(N-1) \times (N-1)$ lattices. In alternating sign matrix (ASM) language this result implies that the doubly refined x-enumerations of ASMs are just appropriate combinations of the singly refined ones.

[1]On leave of absence from St-Petersburg Department of V. A. Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St-Petersburg, Russia.

1. Introduction

The six-vertex model, introduced in [1], and solved for periodic boundary conditions in [2–4], see [5,6] for a review, has turned out to be of great interest also in the case where domain wall boundary conditions (DWBC) are imposed. These boundary conditions were originally introduced for the six-vertex model in the investigation of the norms of Bethe states [7], in the context of the quantum inverse scattering method (QISM) [8]. An important result was obtained in [9] where an exact determinant formula for the partition function was obtained, see also [10]. Subsequently, this result was found of fundamental importance in the proof of long-standing conjectures in enumerative combinatorics, due to the close connection of the model with alternating sign matrices (ASMs) [11–16], see also [17] for a review. It should be mentioned that ASM enumerations appear to be in turn deeply related with quantum spin chains and some loop models, via Razumov-Stroganov conjecture [18]; for recent results, see for instance [19–21] and references therein.

An important information is also contained in correlation functions. However, because of difficulties caused by the lack of translational invariance, their computation is still an open problem in the six-vertex model with DWBC. Some simplifications take place when correlations are considered in vicinity of the boundaries [22]. The simplest one-point boundary correlation functions were investigated in [23], where determinant representations were obtained, analogous to that of papers [9,10] for the partition function. Even if these are almost the simplest correlations one can study for the considered model, they are nevertheless of interest, especially from a combinatorial point of view.

In the present paper we pursue the investigation of the boundary correlation functions. Here we evaluate the two-point boundary correlation function which gives the probability of having particular vertex states at two specific sites in the first and last row of the lattice. Such a correlation function was recently discussed in [24], in the context of a graphical interpretation of the approach of [23]. From a combinatorial point of view this correlation function is closely related with the doubly refined x-enumerations of ASMs; the case of doubly refined 1-enumeration of ASMs was studied previously in [25] where an

explicit expression for the corresponding generating function was derived. An interest in the doubly refined enumerations, in connection with the Razumov-Stroganov conjecture, was recently stressed in [26]. Here we give an explicit, general, and relatively simple expression for these quantities.

More specifically, we show that the mentioned two-point boundary correlation function, for generic values of the six-vertex model weights, is expressible in a very simple way in terms of the analogous one-point boundary correlation functions. This implies that by specializing the parameters of the model to the values corresponding to weighted enumerations of ASMs, one can directly obtain explicit formulae for the doubly refined weighted enumerations of ASMs from our results here. In particular, the doubly refined 1-, 2-, and 3-enumerations of ASMs can be easily found from the corresponding singly refined ones.

To derive the result we consider first the two-point boundary correlation function in the more general case of the inhomogeneous six-vertex model. In this case QISM can be applied. The homogeneous limit is performed next, and a determinant representation, analogous to those given in [9, 10, 23], is derived. Then, using standard techniques of the theory of orthogonal polynomials, along the lines of our recent paper [27], we obtain the final formula for the two-point boundary correlation function.

2. The six-vertex model with DWBC and QISM

In this paper we consider the six-vertex model on an $N \times N$ square lattice with the domain wall boundary conditions (DWBC). Recall that the six-vertex model is a model of arrows residing on the edges of the lattice, with the same number of incoming and outgoing arrows though each lattice vertex (this constraint being known as the 'ice rule'). Each vertex can be in one of six possible states $i = 1, \ldots, 6$, see figure 1. A Boltzmann weight w_i is assigned to each vertex according to its state i; the weights are usually chosen to obey the arrow-reversal symmetry

$$w_1 = w_2 = a, \qquad w_3 = w_4 = b, \qquad w_5 = w_6 = c. \tag{2.1}$$

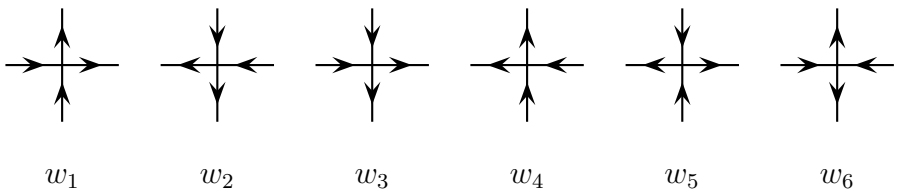

Figure 1: The six states and their Boltzmann weights.

The vertex weights are parameterized in the standard way in terms of a spectral parameter λ and a crossing parameter η,

$$a = \sin(\lambda + \eta), \qquad b = \sin(\lambda - \eta), \qquad c = \sin(2\eta). \tag{2.2}$$

In the case of the $N \times N$ lattice, the ice rule allows imposing domain wall boundary conditions to the six-vertex model. This means fixing the direction of the boundary arrows as follows: all arrows on the left and right boundaries are outgoing while on the top and bottom boundaries they are incoming. The partition function of the model, denoted as Z_N, is the sum over all possible arrow configurations

$$Z_N = \sum a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} \tag{2.3}$$

where n_i denotes the number of vertices of type i in a configuration and n_i's satisfy $n_1 + \cdots + n_6 = N^2$.

To apply the quantum inverse scattering method (QISM) we shall consider the six-vertex model with DWBC in its inhomogeneous version, namely, when the weights of the vertex being at the intersection of k-th horizontal line (row) and α-th vertical line (column) are parameterized as

$$a(\lambda_\alpha, \nu_k) = \sin(\lambda_\alpha - \nu_k + \eta) \qquad b(\lambda_\alpha, \nu_k) = \sin(\lambda_\alpha - \nu_k - \eta) \qquad c(\lambda_\alpha, \nu_k) = c = \sin(2\eta). \tag{2.4}$$

The spectral parameters $\lambda_1, \ldots, \lambda_N$ and $\nu_1, \ldots, \nu_N$ are assumed to be different within each set. A lattice with DWBC, and the assignment of the spectral parameters to rows and columns are shown in figure 2. Note, that we enumerate columns (labelled by Greek indices) from right to left and rows (labelled by Latin indices) from top to bottom.

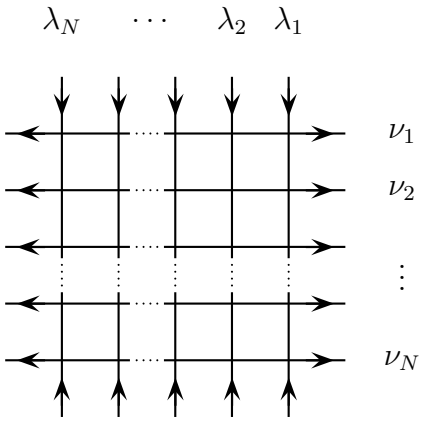

Figure 2: A lattice with DWBC.

Apparently, the partition function (and correlation functions) of the inhomogeneous model is a function of $2N$ variables, $Z_N = Z_N(\{\lambda_\alpha\}, \{\nu_k\})$. After applying QISM the spectral parameters will be set equal within each set: $\lambda_\alpha = \lambda$ and $\nu_k = 0$. We shall refer to this procedure as the homogeneous limit.

We shall now introduce the main objects of QISM, such as L-operator and monodromy matrix. The L-operator of the six-vertex model is nothing but a matrix of the Boltzmann weights. To each vertex being intersection of α-th column and k-th row one can associate the operator $L_{\alpha k}(\lambda_\alpha, \nu_k)$ which acts in the direct product of two vector spaces $\mathbb{C}^2$: in the 'horizontal' space $\mathcal{H}_k = \mathbb{C}^2$ (associated with the k-th row) and in the 'vertical' space $\mathcal{V}_\alpha = \mathbb{C}^2$ (associated with the α-th column). The arrow states on the top and right edges of the vertex can be viewed as 'in' indices of the L-operator while those on the bottom and left edges as 'out' ones. Using spin up and spin down states

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.5}$$

as a basis in $\mathbb{C}^2$, we define L-operator by assuming further that up and right arrows correspond to spin up components while down and left arrows correspond to spin down ones. For the L-operator we have then the expression

$$L_{\alpha k}(\lambda_\alpha, \nu_k) = \sin(\lambda_\alpha - \nu_k + \eta\,\sigma^z_\alpha \sigma^z_k) + \sin(2\eta)(\sigma^-_\alpha \sigma^+_k + \sigma^+_\alpha \sigma^-_k), \tag{2.6}$$

where $\sigma^z, \sigma^\pm = (1/2)(\sigma^x \pm \mathrm{i}\sigma^y)$ are Pauli matrices and the subscripts in (2.6) indicate the space in which they act.

The monodromy matrix is an ordered product of L-operators. We shall define it here as a product along a column, promoting the corresponding vertical space $\mathcal{V}_\alpha$ to be 'auxiliary' space while the horizontal spaces $\mathcal{H}_k$ will be treated as 'quantum' spaces; the space $\mathcal{H} = \otimes_{k=1}^N \mathcal{H}_k$ is thus the total quantum space. To define the monodromy matrix it is more convenient to think of L-operator as acting in $\mathcal{V}_\alpha \otimes \mathcal{H}$, moreover, writing it as 2×2 matrix in $\mathcal{V}_\alpha$ with quantum operator entries acting in $\mathcal{H}$,

$$L_{\alpha k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} \sin(\lambda_\alpha - \nu_k + \eta\,\sigma_k^z) & \sin(2\eta)\,\sigma_k^- \\ \sin(2\eta)\,\sigma_k^+ & \sin(\lambda_\alpha - \nu_k - \eta\,\sigma_k^z) \end{pmatrix}_{[\alpha]}. \tag{2.7}$$

Here $[\alpha]$ indicates that this is the matrix with respect to the auxiliary space $\mathcal{V}_\alpha$ and $\sigma_k^{\pm,z}$ stand for quantum operators in $\mathcal{H}$, acting as Pauli matrices in $\mathcal{H}_k$ and identically elsewhere. The monodromy matrix is defined by

$$T_\alpha(\lambda_\alpha) = \overleftarrow{\prod_{k=1}^{N}} L_{\alpha k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} A(\lambda_\alpha) & B(\lambda_\alpha) \\ C(\lambda_\alpha) & D(\lambda_\alpha) \end{pmatrix}_{[\alpha]}. \tag{2.8}$$

The operators $A(\lambda) = A(\lambda; \nu_1, \ldots, \nu_N)$, etc, act in $\mathcal{H}$ and play a fundamental role in QISM.

These operators, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ are subject to the Yang-Baxter algebra,

$$R_{\alpha\alpha'}(\lambda, \lambda')\big[T_\alpha(\lambda) \otimes T_{\alpha'}(\lambda')\big] = \big[T_\alpha(\lambda') \otimes T_{\alpha'}(\lambda)\big] R_{\alpha\alpha'}(\lambda, \lambda'). \tag{2.9}$$

generated by the six-vertex model R-matrix,

$$R_{\alpha\alpha'}(\lambda, \lambda') = \begin{pmatrix} f(\lambda', \lambda) & 0 & 0 & 0 \\ 0 & g(\lambda', \lambda) & 1 & 0 \\ 0 & 1 & g(\lambda', \lambda) & 0 \\ 0 & 0 & 0 & f(\lambda', \lambda) \end{pmatrix}_{[\alpha\alpha']}. \tag{2.10}$$

where

$$f(\lambda', \lambda) = \frac{\sin(\lambda - \lambda' + 2\eta)}{\sin(\lambda - \lambda')}, \qquad g(\lambda', \lambda) = \frac{\sin(2\eta)}{\sin(\lambda - \lambda')}. \tag{2.11}$$

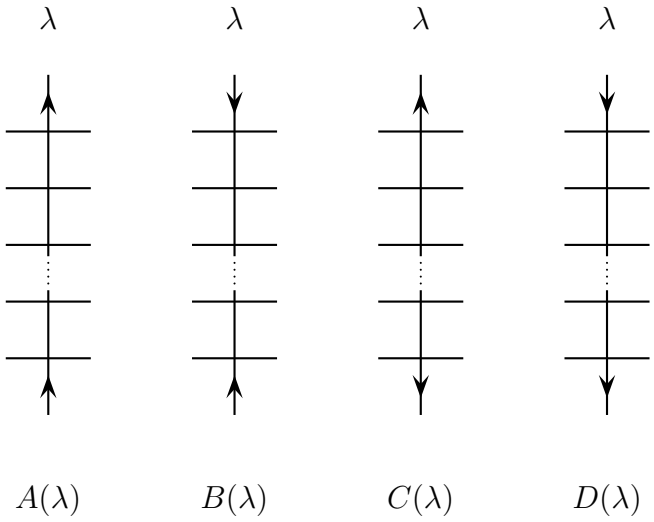

Figure 3: Graphical interpretation of the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$.

The relation (2.9) is also known as RTT relation and it is a consequence of the similar RLL relation. Among the sixteen relations contained in (2.9), in particular, are the following

$$B(\lambda)\,B(\lambda') = B(\lambda')\,B(\lambda), \tag{2.12}$$

$$A(\lambda)\,B(\lambda') = f(\lambda,\lambda')\,B(\lambda')\,A(\lambda) + g(\lambda',\lambda)\,B(\lambda)\,A(\lambda'), \tag{2.13}$$

$$D(\lambda)\,B(\lambda') = f(\lambda',\lambda)\,B(\lambda')\,D(\lambda) + g(\lambda,\lambda')\,B(\lambda)\,D(\lambda'), \tag{2.14}$$

$$C(\lambda)\,B(\lambda') = B(\lambda')\,C(\lambda) + g(\lambda,\lambda')\big[A(\lambda)\,D(\lambda') - A(\lambda')\,D(\lambda)\big], \tag{2.15}$$

which will be used below.

The operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ admit simple graphical interpretation as columns of the lattice, with top and bottom arrows fixed, see figure 3. Taking into account this nice graphical interpretation and exploiting further the correspondence between arrows and spin states to express arrows on the left and right boundaries of the lattice in terms of 'all spins down' and 'all spins up' states,

$$\langle\Downarrow| = \bigotimes_{k=1}^{N} {}_k\langle\downarrow|\,, \qquad |\Uparrow\rangle = \bigotimes_{k=1}^{N} |\uparrow\rangle_k, \tag{2.16}$$

where $|\uparrow\rangle_k$ and $|\downarrow\rangle_k$ are the spin up and spin down vectors of the space $\mathcal{H}_k$, it can be easily seen that the partition function, $Z_N = Z_N(\{\lambda_\alpha\};\{\nu_k\})$, of the six-vertex model

with DWBC is given by

$$Z_N = \langle \Downarrow | B(\lambda_N) \cdots B(\lambda_1) | \Uparrow \rangle. \tag{2.17}$$

Note, that because of relation (2.12) the actual order of operators $B(\lambda)$ is not important in this formula.

The explicit expression for the partition function, found and proven in [9], see also [10], reads

$$Z_N = \frac{\prod_{\alpha=1}^{N} \prod_{k=1}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)}{\prod_{1 \le \alpha < \beta \le N} d(\lambda_\beta, \lambda_\alpha) \prod_{1 \le j < k \le N} d(\nu_j, \nu_k)} \det_N T \tag{2.18}$$

where

$$d(\lambda, \lambda') := \sin(\lambda - \lambda') \tag{2.19}$$

and the functions $a(\lambda, \nu)$ and $b(\lambda, \nu)$ are defined in (2.2). The $N \times N$ matrix T is given by

$$T_{\alpha k} = t(\lambda_\alpha, \nu_k), \qquad t(\lambda, \nu) = \frac{\sin(2\eta)}{\sin(\lambda - \nu + \eta) \sin(\lambda - \nu - \eta)}. \tag{2.20}$$

In [9, 10] the determinant formula (2.18) was proven on the basis of certain recursion formulae, established in [7]. Another proof, solely on the basis of Yang-Baxter algebra, was given in [23]; in the next section we sketch the derivation.

In the homogenous limit, when $\lambda_\alpha = \lambda$ and $\nu_k = 0$, expression (2.18) turns into the following one [10]

$$Z_N = \frac{[\sin(\lambda - \eta) \sin(\lambda + \eta)]^{N^2}}{\prod_{n=1}^{N-1} (n!)^2} \det_N \Phi \tag{2.21}$$

where

$$\Phi_{\alpha k} = \partial_\lambda^{\alpha + k - 2} \varphi(\lambda, \eta), \qquad \varphi(\lambda, \eta) = \frac{\sin(2\eta)}{\sin(\lambda - \eta) \sin(\lambda + \eta)}. \tag{2.22}$$

The procedure of obtaining (2.21) from (2.18) was explained in detail in [10]. In our treatment of the correlation functions we shall proceed in the same way, first obtaining an expression for the inhomogeneous model, and next taking the homogenous limit.

3. One-point boundary correlation functions

Here we recall the main results of paper [23] on one-point boundary correlation functions. In the next section we shall explain how this approach can be used to compute two-point

boundary correlation functions.

In paper [23] two closely related kinds of one-point boundary correlation functions were considered. The first correlation function, denoted as $H_N^{(r)}$, is the probability of finding the r-th vertex (counted from the right) on the first row in the state $i = 5$. Formally, this correlation function can be defined by

$$H_N^{(r)} = Z_N^{-1} \langle \Downarrow | B(\lambda_N) \dots B(\lambda_{r+1})\, q_1\, B(\lambda_r)\, p_1\, B(\lambda_{r-1}) \cdots B(\lambda_1) | \Uparrow \rangle \tag{3.1}$$

where q_k and p_k denote projection operators on the spin up and spin down states, at k-th 'site', respectively,

$$q_k = \frac{1}{2}(1 - \sigma_k^z), \qquad p_k = \frac{1}{2}(1 + \sigma_k^z). \tag{3.2}$$

The second correlation, denoted as $G_N^{(r)}$, is the boundary 'polarization', i.e. the probability of finding an arrow pointing left on the horizontal edge of the first row between r-th and $(r+1)$-th columns. One can define this correlation function by

$$G_N^{(r)} = Z_N^{-1} \langle \Downarrow | B(\lambda_N) \dots B(\lambda_{r+1})\, q_1\, B(\lambda_r) \cdots B(\lambda_1) | \Uparrow \rangle. \tag{3.3}$$

Due to DWBC, the two correlation function $H_N^{(r)}$ and $G_N^{(r)}$ are related by

$$G_N^{(r)} = \sum_{\alpha=1}^{r} H_N^{(\alpha)}; \qquad H_N^{(r)} = G_N^{(r)} - G_N^{(r-1)}. \tag{3.4}$$

These relations can be easily found by exploring the graphical interpretation of these functions, see figure 4.

To compute the one-point boundary correlation functions one can introduce the operators $\widetilde{A}(\lambda)$, $\widetilde{B}(\lambda)$, $\widetilde{C}(\lambda)$ and $\widetilde{D}(\lambda)$ as entries of the monodromy matrix on $N-1$ sites,

$$\widetilde{T}_\alpha(\lambda_\alpha) := \overleftarrow{\prod_{k=2}^{N}} L_{\alpha,k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} \widetilde{A}(\lambda_\alpha) & \widetilde{B}(\lambda_\alpha) \\ \widetilde{C}(\lambda_\alpha) & \widetilde{D}(\lambda_\alpha) \end{pmatrix}. \tag{3.5}$$

Correspondingly, one can define

$$\langle \widetilde{\Downarrow} | = \bigotimes_{k=2}^{N} {}_k\langle \uparrow |, \qquad | \widetilde{\Uparrow} \rangle = \bigotimes_{k=2}^{N} | \uparrow \rangle_k. \tag{3.6}$$

One has

$$\widetilde{A}(\lambda) | \widetilde{\Uparrow} \rangle = \prod_{k=2}^{N} a(\lambda, \nu_k) | \widetilde{\Uparrow} \rangle, \qquad \widetilde{D}(\lambda) | \widetilde{\Uparrow} \rangle = \prod_{k=2}^{N} b(\lambda, \nu_k) | \widetilde{\Uparrow} \rangle. \tag{3.7}$$

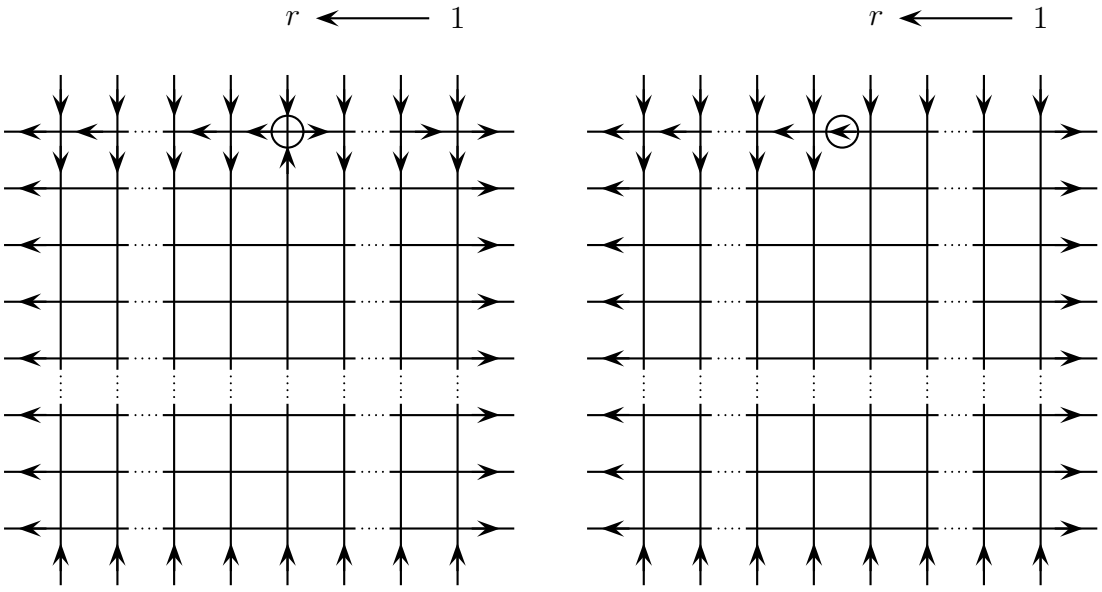

Figure 4: Boundary one-point correlation functions: (a) function $H_N^{(r)}$; (b) function $G_N^{(r)}$.

In what follows, to simplify writing, we shall omit tildes over operators and vectors.

The role of these new operators in application to the one-point boundary correlators is quite clear from their graphical interpretation shown on figure 3 and the analogous graphical representation for the correlation functions shown on figure 4. Specifically, in the case of $H_N^{(r)}$ all vertices of the first row are fixed, thus leading to the following formula in terms of the operators on $N-1$ sites

$$H_N^{(r)} = Z_N^{-1}\, c \prod_{\alpha=r+1}^{N} a(\lambda_\alpha,\nu_1) \prod_{\alpha=1}^{r-1} b(\lambda_\alpha,\nu_1) \times \langle \Downarrow | B(\lambda_N)\cdots B(\lambda_{r+1})\, A(\lambda_r)\, B(\lambda_{r-1})\cdots B(\lambda_1) | \Uparrow \rangle. \quad (3.8)$$

Similarly, the correlation function $G_N^{(r)}$ can be written as a sum of r such terms

$$G_N^{(r)} = Z_N^{-1}\, c \sum_{\alpha=1}^{r} \prod_{\beta=\alpha+1}^{N} a(\lambda_\beta,\nu_1) \prod_{\beta=1}^{\alpha-1} b(\lambda_\beta,\nu_1) \times \langle \Downarrow | B(\lambda_N)\cdots B(\lambda_{\alpha+1})\, A(\lambda_\alpha)\, B(\lambda_{\alpha-1})\cdots B(\lambda_1) | \Uparrow \rangle. \quad (3.9)$$

On the first stage of computation the correlation function are expressed in terms of

the partition functions on $(N-1)\times(N-1)$ sublattices. Using the commutation relation (2.13) we have

$$A(\lambda_r)\prod_{\alpha=1}^{r-1}B(\lambda_\alpha)|\Uparrow\rangle=\sum_{\alpha=1}^{r}\prod_{k=2}^{N}a(\lambda_\alpha,\nu_k)\frac{g(\lambda_\alpha,\lambda_r)}{f(\lambda_\alpha,\lambda_r)}\prod_{\substack{\beta=1\\\beta\neq\alpha}}^{r}f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\beta=1\\\beta\neq\alpha}}^{r}B(\lambda_\beta)|\Uparrow\rangle. \quad (3.10)$$

The last relation allows one to obtain, in the case of $H_N^{(r)}$, the expression

$$H_N^{(r)}=Z_N^{-1}\,c\prod_{\alpha=r+1}^{N}a(\lambda_\alpha,\nu_1)\prod_{\alpha=1}^{r-1}b(\lambda_\alpha,\nu_1)$$
$$\times\sum_{\beta=1}^{r}\prod_{k=2}^{N}a(\lambda_\beta,\nu_k)\frac{g(\lambda_\beta,\lambda_r)}{f(\lambda_\beta,\lambda_r)}\prod_{\substack{\gamma=1\\\gamma\neq\beta}}^{r}f(\lambda_\beta,\lambda_\gamma)Z_{N-1}\big(\{\lambda_\delta\}_{\delta=1,\delta\neq\alpha}^{N};\{\nu_k\}_{k=2}^{N}\big) \quad (3.11)$$

The corresponding expression for $G_N^{(r)}$ can be immediately obtained using the relationship with the function $H_N^{(r)}$, see (3.4). However, as explained in [23], in this case one gets the result in terms of some double sum which is not actually an analogue of (3.11). It was pointed out that the analogue of (3.11) can be found if one uses instead the relation

$$\sum_{\alpha=1}^{r}\prod_{\beta=\alpha+1}^{r}a(\lambda_\beta,\nu_1)\prod_{\beta=1}^{\alpha-1}b(\lambda_\beta,\nu_1)\;B(\lambda_r)\cdots B(\lambda_{\alpha+1})A(\lambda_\alpha)B(\lambda_{\alpha-1})\cdots B(\lambda_1)|\Uparrow\rangle$$
$$=\sum_{\alpha=1}^{r}\prod_{k=2}^{N}a(\lambda_\alpha,\nu_k)\prod_{\substack{\beta=1\\\beta\neq\alpha}}^{r}b(\lambda_\beta,\nu_1)\prod_{\substack{\beta=1\\\beta\neq\alpha}}^{r}f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\beta=1\\\beta\neq\alpha}}^{r}B(\lambda_\beta)|\Uparrow\rangle \quad (3.12)$$

which can be obtained by means of (2.13) by taking into account that LHS here is symmetric with respect to permutations of $\lambda_1,\ldots,\lambda_r$ by construction (for more details on derivation of such relations see, e.g., [8], sections VII.2 and XXII.2). In this way one obtains

$$G_N^{(r)}=Z_N^{-1}\prod_{\alpha=r+1}^{N}a(\lambda_\alpha,\nu_1)\prod_{\alpha=1}^{r}b(\lambda_\alpha,\nu_1)$$
$$\times\sum_{\beta=1}^{r}\frac{c}{b(\lambda_\beta,\nu_1)}\prod_{k=2}^{N}a(\lambda_\beta,\nu_k)\prod_{\substack{\gamma=1\\\gamma\neq\beta}}^{r}f(\lambda_\beta,\lambda_\gamma)Z_{N-1}\big(\{\lambda_\delta\}_{\delta=1,\delta\neq\beta}^{N};\{\nu_k\}_{k=2}^{N}\big). \quad (3.13)$$

As explained in paper [23] this representation is important since it allows one to prove the determinant formula (2.18). Indeed, since by definition $G_N^{(N)}=1$, expression (3.13)

turns into some recurrence relation connecting the partition functions Z_N and Z_{N-1}; note, that here all the parameters in the sets $\{\lambda_\alpha\}_{\alpha=1}^N$ and $\{\nu_k\}_{k=1}^N$ are assumed to be completely arbitrary (cf. [9, 10]). As explained in [23], it can be readily shown, in virtue of the Kramer rule and of some particular identity, that the solution of this recurrence relation, with the initial condition $Z_1 = c$, is given by determinant formula (2.18).

Thus, the expression for the partition function being proved within the considered framework, it can be used to obtain similar representations for one-point functions. Substituting the expression for Z_{N-1} in (3.11) and (3.13) gives rise to the determinant formulae for one-point functions, see again [23] for details. We end up this section by quoting the results.

The function $H_N^{(r)}$ is given by

$$H_N^{(r)} = \frac{c\prod_{k=2}^N d(\nu_1,\nu_k)}{\prod_{\alpha=1}^r a(\lambda_\alpha,\nu_1)\prod_{\alpha=r}^N b(\lambda_\alpha,\nu_1)}\,\frac{\det_N V}{\det_N T} \tag{3.14}$$

where the matrix V differs from the matrix T, equation (2.20), just by the elements of the first column,

$$V_{\alpha,1} = v_r(\lambda_\alpha); \qquad V_{\alpha,k} = T_{\alpha,k}, \quad k = 2,\dots,N. \tag{3.15}$$

Here the function $v_r(\lambda)$ is given by

$$v_r(\lambda) = \frac{\prod_{\alpha=r+1}^N d(\lambda_\alpha,\lambda)\prod_{\alpha=1}^{r-1} e(\lambda_\alpha,\lambda)}{\prod_{k=2}^N b(\lambda,\nu_k)} \tag{3.16}$$

where

$$e(\lambda,\lambda') = \sin(\lambda-\lambda'+2\eta) \tag{3.17}$$

and the function $d(\lambda,\lambda')$ is defined in (2.19).

The function $G_N^{(r)}$ is given by

$$G_N^{(r)} = \frac{\prod_{k=2}^N d(\nu_1,\nu_k)}{\prod_{\alpha=1}^r a(\lambda_\alpha,\nu_1)\prod_{\alpha=r+1}^N b(\lambda_\alpha,\nu_1)}\,\frac{\det_N S}{\det_N T} \tag{3.18}$$

where the matrix S also differs from T just by the elements of the first column,

$$S_{\alpha,1} = s_r(\lambda_\alpha); \qquad S_{\alpha,k} = T_{\alpha,k}, \quad k = 2,\dots,N. \tag{3.19}$$

Here the function $s_r(\lambda)$ is given by

$$s_r(\lambda) = \frac{\prod_{\alpha=r+1}^{N} d(\lambda_\alpha, \lambda) \prod_{\alpha=1}^{r} e(\lambda_\alpha, \lambda)}{\prod_{k=1}^{N} b(\lambda, \nu_k)}. \tag{3.20}$$

In the homogeneous limit the following formulae are valid. The function $H_N^{(r)}$ is given by

$$H_N^{(r)} = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r \big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\det_N \Psi}{\det_N \Phi} \tag{3.21}$$

where the matrix Ψ differs from the matrix Φ, equation (2.22), just by the elements of the last column

$$\Psi_{\alpha,k} = \Phi_{\alpha,k}, \quad k = 1, \dots, N-1; \qquad \Psi_{\alpha,N} = \partial_\varepsilon^{\alpha-1} \frac{(\sin\varepsilon)^{N-r}[\sin(\varepsilon-2\eta)]^{r-1}}{[\sin(\varepsilon+\lambda-\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \tag{3.22}$$

Similarly, the function $G_N^{(r)}$ is given by

$$G_N^{(r)} = \frac{(N-1)!}{\big[\sin(\lambda+\eta)\big]^r \big[\sin(\lambda-\eta)\big]^{N-r}}\,\frac{\det_N \Theta}{\det_N \Phi} \tag{3.23}$$

where

$$\Theta_{\alpha,k} = \Phi_{\alpha,k}, \quad k = 1, \dots, N-1; \qquad \Theta_{\alpha,N} = -\partial_\varepsilon^{\alpha-1} \frac{(\sin\varepsilon)^{N-r}[\sin(\varepsilon-2\eta)]^{r}}{[\sin(\varepsilon+\lambda-\eta)]^{N}}\bigg|_{\varepsilon=0}. \tag{3.24}$$

4. Two-point boundary correlations

The one-point boundary correlation functions just considered can be directly generalized to the case of two-point ones [24]. Here we shall consider in detail derivation of the function $H_N^{(r_1,r_2)}$, which gives the probability of finding vertices of type $i = 5$ on the opposite, top and bottom, boundaries. More precisely, we define $H_N^{(r_1,r_2)}$ as the probability to find vertices of type $i = 5$ both at r_1-th position (counted from the right) of the first row and at r_2-th position of the last row, see figure 5. Formally, if $r_1 < r_2$ then one can define this correlation function by

$$H_N^{(r_1,r_2)} = Z_N^{-1} \langle \Downarrow | B(\lambda_N) \cdots B(\lambda_{r_2+1})\, q_N\, B(\lambda_{r_2})\, p_N\, B(\lambda_{r_2-1}) \cdots \\ \times \cdots B(\lambda_{r_1+1})\, q_1\, B(\lambda_{r_1})\, p_1\, B(\lambda_{r_1-1}) \cdots B(\lambda_1) | \Uparrow \rangle. \tag{4.1}$$

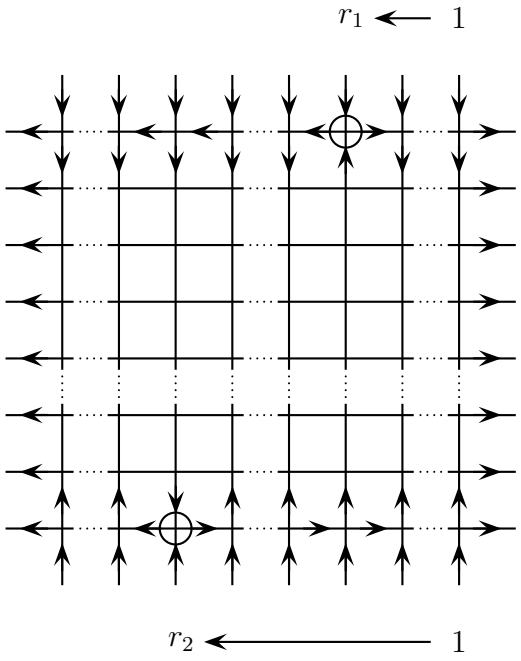

Figure 5: The two-point correlation function $H_N^{(r_1,r_2)}$.

For $r_1 = r_2$ or $r_1 > r_2$ one has analogous expressions.

Quite similarly, one can also consider the function $G_N^{(r_1,r_2)}$, giving the probability of finding arrows pointing left on the first and the last rows. The functions $H_N^{(r_1,r_2)}$ and $G_N^{(r_1,r_2)}$ are related to each other just as in the one-point case; here we focus on the function $H_N^{(r_1,r_2)}$ since it is the most interesting for combinatorial applications. The result for $G_N^{(r_1,r_2)}$ can be found using its obvious relation with $H_N^{(r_1,r_2)}$.

To compute the function $H_N^{(r_1,r_2)}$ we introduce the operators $\widetilde{A}(\lambda)$, $\widetilde{B}(\lambda)$, $\widetilde{C}(\lambda)$ and $\widetilde{D}(\lambda)$ as entries of the monodromy matrix on $N-2$ sites,

$$\widetilde{T}_\alpha(\lambda_\alpha) := \prod_{k=2}^{\overleftarrow{N-1}} L_{\alpha,k}(\lambda_\alpha,\nu_k) = \begin{pmatrix} \widetilde{A}(\lambda_\alpha) & \widetilde{B}(\lambda_\alpha) \\ \widetilde{C}(\lambda_\alpha) & \widetilde{D}(\lambda_\alpha) \end{pmatrix}. \tag{4.2}$$

Correspondingly, we define

$$\langle \widetilde{\Downarrow} | = \bigotimes_{k=2}^{N-1} {}_k\langle \uparrow |, \qquad |\widetilde{\Uparrow}\rangle = \bigotimes_{k=2}^{N-1} |\uparrow\rangle_k . \tag{4.3}$$

We have

$$\widetilde{A}(\lambda)|\widetilde{\Uparrow}\rangle = \prod_{k=2}^{N-1} a(\lambda,\nu_k)|\widetilde{\Uparrow}\rangle, \qquad \widetilde{D}(\lambda)|\widetilde{\Uparrow}\rangle = \prod_{k=2}^{N-1} b(\lambda,\nu_k)|\widetilde{\Uparrow}\rangle. \tag{4.4}$$

In what follows, to simplify writing, we shall again omit tildes over operators and vectors.

On the first stage of computation we express the correlation function $H_N^{(r_1,r_2)}$ in terms of partition functions on $(N-2)\times(N-2)$ lattices. This can be done by directly following the procedure of the previous section. For an alternative way of performing this preliminary step, see also [24].

Let us assume that $r_1 \neq r_2$. If $r_1 < r_2$ then, using the same arguments as for one-point correlations (e.g., by invoking the graphical interpretation of the operators shown on figure 3), for the two-point correlator $H_N^{(r_1,r_2)}$ we write

$$H_N^{(r_1,r_2)} = Z_N^{-1}\, c^2 \prod_{\alpha=r_1+1}^{N} a(\lambda_\alpha,\nu_1) \prod_{\alpha=1}^{r_1-1} b(\lambda_\alpha,\nu_1) \prod_{\alpha=r_2+1}^{N} b(\lambda_\alpha,\nu_N) \prod_{\alpha=1}^{r_2-1} a(\lambda_\alpha,\nu_N)$$
$$\times \langle \Downarrow | B(\lambda_N)\cdots B(\lambda_{r_2+1})\, D(\lambda_{r_2})\, B(\lambda_{r_2-1})\cdots B(\lambda_{r_1+1})\, A(\lambda_{r_1})\, B(\lambda_{r_1-1})\cdots B(\lambda_1) | \Uparrow \rangle. \tag{4.5}$$

Acting first with the operator $A(\lambda_{r_1})$ on the right, using

$$A(\lambda_r)\prod_{\alpha=1}^{r-1} B(\lambda_\alpha)|\Uparrow\rangle = \sum_{\alpha=1}^{r}\prod_{k=2}^{N-1} a(\lambda_\alpha,\nu_k)\frac{g(\lambda_\alpha,\lambda_r)}{f(\lambda_\alpha,\lambda_r)}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r} f(\lambda_\alpha,\lambda_\beta)\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r} B(\lambda_\beta)|\Uparrow\rangle, \tag{4.6}$$

and next with the operator $D(\lambda_{r_2})$, using

$$D(\lambda_r)\prod_{\alpha=1}^{r-1} B(\lambda_\alpha)|\Uparrow\rangle = \sum_{\alpha=1}^{r}\prod_{k=2}^{N-1} b(\lambda_\alpha,\nu_k)\frac{g(\lambda_r,\lambda_\alpha)}{f(\lambda_r,\lambda_\alpha)}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r} f(\lambda_\beta,\lambda_\alpha)\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{r} B(\lambda_\beta)|\Uparrow\rangle, \tag{4.7}$$

we straightforwardly obtain

$$H_N^{(r_1,r_2)} = Z_N^{-1}\, c^2 \prod_{\alpha=r_1+1}^{N} a(\lambda_\alpha,\nu_1) \prod_{\alpha=1}^{r_1-1} b(\lambda_\alpha,\nu_1) \prod_{\alpha=r_2+1}^{N} b(\lambda_\alpha,\nu_N) \prod_{\alpha=1}^{r_2-1} a(\lambda_\alpha,\nu_N)$$
$$\times \sum_{\alpha=1}^{r_1}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{r_2}\prod_{k=2}^{N-1} a(\lambda_\alpha,\nu_k)\frac{g(\lambda_\alpha,\lambda_{r_1})}{f(\lambda_\alpha,\lambda_{r_1})}\prod_{\substack{\gamma=1\\ \gamma\neq\alpha}}^{r_1} f(\lambda_\alpha,\lambda_\gamma)\prod_{k=2}^{N-1} b(\lambda_\beta,\nu_k)\frac{g(\lambda_{r_2},\lambda_\beta)}{f(\lambda_{r_2},\lambda_\beta)}\prod_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{r_2} f(\lambda_\gamma,\lambda_\beta)$$
$$\times Z_{N-2}\Big(\{\lambda_\delta\}_{\delta=1,\delta\neq\alpha,\beta}^{N};\{\nu_k\}_{k=2}^{N-1}\Big). \tag{4.8}$$

If $r_1 > r_2$ then one should first act with the operator $D(\lambda_{r_2})$, and next with the operator $A(\lambda_{r_1})$. It can be easily verified that one obtains exactly the same expression, i.e., formula (4.8) is valid for $r_1 \neq r_2$.

Consider now the case $r_1 = r_2 = r$. In this case instead of (4.5) we have

$$H_N^{(r,r)} = Z_N^{-1}\, c^2 \prod_{\alpha=r+1}^{N} a(\lambda_\alpha,\nu_1)\, b(\lambda_\alpha,\nu_N) \prod_{\alpha=1}^{r-1} b(\lambda_\alpha,\nu_1)\, a(\lambda_\alpha,\nu_N)$$
$$\times \langle \Downarrow | B(\lambda_N)\cdots B(\lambda_{r+1})\, C(\lambda_r)\, B(\lambda_{r-1})\cdots B(\lambda_1) | \Uparrow \rangle. \quad (4.9)$$

Using commutation relation (2.15) the following formula can be derived (see [8], section VII.2.2)

$$C(\lambda_r) \prod_{\alpha=1}^{r-1} B(\lambda_\alpha) | \Uparrow \rangle = \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{r} \prod_{k=2}^{N-1} a(\lambda_\alpha,\nu_k) \prod_{k=2}^{N-1} b(\lambda_\beta,\nu_k)$$
$$\times \frac{g(\lambda_\alpha,\lambda_r) g(\lambda_r,\lambda_\beta)}{f(\lambda_\alpha,\lambda_r) f(\lambda_r,\lambda_\beta)} f(\lambda_\alpha,\lambda_\beta) \prod_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^{r} f(\lambda_\alpha,\lambda_\gamma) f(\lambda_\gamma,\lambda_\beta) \prod_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^{r} B(\lambda_\gamma) | \Uparrow \rangle. \quad (4.10)$$

Thus, in this case we have

$$H_N^{(r,r)} = Z_N^{-1}\, c^2 \prod_{\alpha=r+1}^{N} a(\lambda_\alpha,\nu_1)\, b(\lambda_\alpha,\nu_N) \prod_{\alpha=1}^{r-1} b(\lambda_\alpha,\nu_1)\, a(\lambda_\alpha,\nu_N)$$
$$\times \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{r} \prod_{k=2}^{N-1} a(\lambda_\alpha,\nu_k) \prod_{k=2}^{N-1} b(\lambda_\beta,\nu_k) \frac{g(\lambda_\alpha,\lambda_r) g(\lambda_r,\lambda_\beta)}{f(\lambda_\alpha,\lambda_r) f(\lambda_r,\lambda_\beta)} f(\lambda_\alpha,\lambda_\beta)$$
$$\times \prod_{\substack{\gamma=1 \\ \gamma\neq\alpha,\beta}}^{r} f(\lambda_\alpha,\lambda_\gamma) f(\lambda_\gamma,\lambda_\beta) Z_{N-2}\Big(\{\lambda_\delta\}_{\delta=1,\delta\neq\alpha,\beta}^{N}; \{\nu_k\}_{k=2}^{N-1} \Big). \quad (4.11)$$

This formula shows that expression (4.8) is, in fact, valid for all values of r_1 and r_2, $1 \leq r_1, r_2 \leq N$ with no further restriction.

Substituting now the determinant representation (2.18) for the partition function Z_N and that for the 'reduced' partition function Z_{N-2} into (4.8), and cancelling whatever possible, we find

$$H_N^{(r_1,r_2)} = \frac{c^2 d(\nu_1,\nu_N) \prod_{k=2}^{N-1} d(\nu_1,\nu_k) d(\nu_k,\nu_N)}{\prod_{\alpha=1}^{r_1} a(\lambda_\alpha,\nu_1) \prod_{\alpha=r_1}^{N} b(\lambda_\alpha,\nu_1) \prod_{\alpha=1}^{r_2} b(\lambda_\alpha,\nu_N) \prod_{\alpha=r_2}^{N} a(\lambda_\alpha,\nu_N)\, \det_N T}$$
$$\times \sum_{\alpha=1}^{r_1} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{r_2} (-1)^{N+\alpha+\beta} \epsilon_{\alpha\beta} \frac{w_{r_1}(\lambda_\alpha) \tilde{w}_{r_2}(\lambda_\beta)}{e(\lambda_\beta,\lambda_\alpha)}\, \det_{N-2} T_{(\alpha,\beta;1,N)}. \quad (4.12)$$

Here $T_{(\alpha,\beta;1,N)}$ denotes the $(N-2)\times(N-2)$ matrix obtained from T by eliminating the α-th and β-th rows and the first and the last columns (i.e., its determinant is just the corresponding minor of T). The symbol $\epsilon_{\alpha\beta}$ stands for the sign function

$$\epsilon_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha = \beta \\ -1 & \text{if } \alpha < \beta \end{cases}. \tag{4.13}$$

The functions $w_r(\lambda)$ and $\tilde{w}(\lambda)$ are defined as (compare with (3.16))

$$w_r(\lambda) = \frac{\prod_{\alpha=r+1}^{N} d(\lambda_\alpha,\lambda)\prod_{\alpha=1}^{r-1} e(\lambda_\alpha,\lambda)}{\prod_{k=2}^{N-1} b(\lambda,\nu_k)}, \tag{4.14}$$

$$\tilde{w}_r(\lambda) = \frac{\prod_{\alpha=r+1}^{N} d(\lambda,\lambda_\alpha)\prod_{\alpha=1}^{r-1} e(\lambda,\lambda_\alpha)}{\prod_{k=2}^{N-1} a(\lambda,\nu_k)}. \tag{4.15}$$

Note, that both $w_r(\lambda_\alpha)$ and $\tilde{w}_r(\lambda_\alpha)$ vanish if $r<\alpha$; thus the double sum in (4.12) can be formally extended to N.

Consider now how the homogeneous limit can be performed in the just obtained expression for the two-point correlation function. Here it should be mentioned that, contrary to the formulae for the partition function and one-point correlators, the two-point boundary correlation function cannot be expressed as a determinant in the inhomogeneous model. Nevertheless, as we shall show now this is just a minor inconvenience which can be very efficiently solved without much efforts.

First of all we mention that the problem we are actually facing is that of finding the homogeneous limit for the following double sum

$$W = \sum_{\alpha,\beta=1}^{N} (-1)^{N+\alpha+\beta}\epsilon_{\alpha\beta}F(\lambda_\alpha,\lambda_\beta)\,\det_{N-2} T_{(\alpha,\beta;1,N)} \tag{4.16}$$

where $F(\lambda,\lambda')$ is some function. Consider the substitution

$$\lambda_\alpha = \lambda + z_\alpha, \qquad \alpha = 1,\ldots,N. \tag{4.17}$$

In what follows we shall send all z_α (and ν_k) to zero but before performing the limit we note that the double sum above can be rewritten, using

$$F(\lambda+z_1,\lambda+z_2) = \Big[\exp(z_\alpha\partial_{\varepsilon_1})\exp(z_\beta\partial_{\varepsilon_2})F(\lambda+\varepsilon_1,\lambda+\varepsilon_2)\Big]\Big|_{\substack{\varepsilon_1=0\\ \varepsilon_2=0}}, \tag{4.18}$$

in a quite formal way as follows

$$W = \begin{vmatrix} \exp(z_1\partial_{\varepsilon_1}) & t(\lambda+z_1,\nu_2) & \dots & t(\lambda+z_1,\nu_{N-1}) & \exp(z_1\partial_{\varepsilon_2}) \\ \exp(z_2\partial_{\varepsilon_1}) & t(\lambda+z_2,\nu_2) & \dots & t(\lambda+z_2,\nu_{N-1}) & \exp(z_2\partial_{\varepsilon_2}) \\ \dots & \dots & \dots & \dots & \dots \\ \exp(z_N\partial_{\varepsilon_1}) & t(\lambda+z_N,\nu_2) & \dots & t(\lambda+z_N,\nu_{N-1}) & \exp(z_N\partial_{\varepsilon_2}) \end{vmatrix} F(\lambda+\varepsilon_1,\lambda+\varepsilon_2)\Big|_{\substack{\varepsilon_1=0\\ \varepsilon_2=0}} \tag{4.19}$$

so that the homogenous limit of expression (4.12) can be performed exactly along the lines given in [10].

Namely, we shall put ν_k's and z_α's to zero in the order $\nu_1,\dots,\nu_N,z_1,\dots,z_N$ each time keeping the leading order. The prefactor in (4.12) becomes

$$\frac{\sin^2(2\eta)}{[\sin(\lambda+\eta)]^{N+r_1-r_2+1}[\sin(\lambda-\eta)]^{N+r_2-r_1+1}\det_N\Phi} \times \frac{(N-1)!\,(N-2)!}{\left(z_2\cdot\frac{z_3^2}{2!}\cdot\frac{z_4^3}{3!}\cdots\frac{z_N^{N-1}}{(N-1)!}\right)\left(\nu_3\cdot\frac{\nu_4^2}{2!}\cdots\frac{\nu_{N-1}^{N-2}}{(N-3)!}\right)} \tag{4.20}$$

while the double sum goes into

$$\left(z_2\cdot\frac{z_3^2}{2!}\cdot\frac{z_4^3}{3!}\cdots\frac{z_N^{N-1}}{(N-1)!}\right)\left(\nu_3\cdot\frac{\nu_4^2}{2!}\cdots\frac{\nu_{N-1}^{N-2}}{(N-3)!}\right) \times \det\left(\Phi_{\alpha,k}\Big|\partial_{\varepsilon_2}^{\alpha-1}\Big|\partial_{\varepsilon_1}^{\alpha-1}\right)_{1\le\alpha\le N,1\le k\le N-2} h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2)\Big|_{\varepsilon_1=\varepsilon_2=0} \tag{4.21}$$

where

$$h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2) = \frac{(\sin\varepsilon_1)^{N-r_1}[\sin(\varepsilon_1-2\eta)]^{r_1-1}(\sin\varepsilon_2)^{N-r_2}[\sin(\varepsilon_2+2\eta)]^{r_2-1}}{\sin(\varepsilon_2-\varepsilon_1+2\eta)[\sin(\varepsilon_1+\lambda-\eta)]^{N-2}[\sin(\varepsilon_2+\lambda+\eta)]^{N-2}}. \tag{4.22}$$

Thus, in the homogeneous limit for the two-point correlation function we obtain

$$H_N^{(r_1,r_2)} = \frac{(N-1)!\,(N-2)!\,\sin^2(2\eta)}{\big[\sin(\lambda+\eta)\big]^{N+r_1-r_2+1}\big[\sin(\lambda-\eta)\big]^{N+r_2-r_1+1}\det_N\Phi} \times \left[\det\left(\Phi_{\alpha,k}\Big|\partial_{\varepsilon_2}^{\alpha-1}\Big|\partial_{\varepsilon_1}^{\alpha-1}\right)_{1\le\alpha\le N,1\le k\le N-2} h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2)\right]\Big|_{\varepsilon_1=\varepsilon_2=0}. \tag{4.23}$$

This determinant representation is analogous to those of the previous section for one-point correlation functions.

5. Orthogonal polynomials representation

In this section the results for one- and two-point boundary correlation functions will be analyzed by making use of the orthogonal polynomials theory, along the lines proposed in paper [27]. Here we show that the two-point boundary correlation function, studied in the previous section, is expressible in terms of one-point ones.

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of polynomials, with non-vanishing leading coefficient

$$P_n(x) = \kappa_n x^n + \ldots, \qquad kappa_n \neq 0, \tag{5.1}$$

and orthogonal on the real axis with respect to some weight $\mu(x)$,

$$\int_{-\infty}^{\infty} P_{n_1}(x)\, P_{n_2}(x)\, \mu(x)\, \mathrm{d}x = h_{n_1}\delta_{n_1 n_2}. \tag{5.2}$$

Let c_n denote n-th moment of the weight $\mu(x)$, i.e.

$$c_n = \int_{-\infty}^{\infty} x^n \mu(x) \mathrm{d}x, \qquad n = 0, 1, \ldots \tag{5.3}$$

and let us consider the $(n+1) \times (n+1)$ determinant

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n} \end{vmatrix}. \tag{5.4}$$

Using the orthogonality condition (5.2) and well-known properties of determinants, one can easily find the following formula

$$\Delta_n = \prod_{k=0}^{n} \frac{h_k}{\kappa_k^2}. \tag{5.5}$$

This formula can be used for computation of determinants, provided the orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are known. On the other hand, the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ can in turn be expressed as determinants. For later use let us introduce the notation

$$\Delta_n^{(k)}(x_1, \ldots, x_k) = \begin{vmatrix} c_0 & c_1 & \ldots & c_{n-k} & 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_{n-k+1} & x_1 & x_2 & \ldots & x_k \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & c_{n+1} & \ldots & c_{2n-k} & x_1^n & x_2^n & \ldots & x_k^n \end{vmatrix} \tag{5.6}$$

so that $\Delta_n^{(0)} \equiv \Delta_n$. For the polynomials one can find that

$$P_n(x) = \frac{\kappa_n}{\Delta_{n-1}} \Delta_n^{(1)}(x). \tag{5.7}$$

For a proof, see, e.g., book [29].

The relation (5.7) can be read off inversely thus giving an expression for the determinant $\Delta_n^{(1)}(x)$ in terms of the polynomials $P_n(x)$. Taking into account that (see (5.5))

$$\frac{h_n}{\kappa_n^2} = \frac{\Delta_n}{\Delta_{n-1}} \tag{5.8}$$

we can write

$$\frac{\Delta_n^{(1)}(x)}{\Delta_n} = \frac{\kappa_n}{h_n}\, P_n(x). \tag{5.9}$$

Consider now the case of $\Delta_n^{(2)}(x_1, x_2)$. It is clear that the term of the highest powers on both x_1 and x_2 is just $\Delta_{n-2}(x_2^n x_1^{n-1} - x_1^n x_2^{n-1})$; extending further the methods of [29] we obtain

$$\Delta_n^{(2)}(x_1, x_2) = \frac{\Delta_{n-2}}{\kappa_n \kappa_{n-1}} \big[P_{n-1}(x_1) P_n(x_2) - P_n(x_1) P_{n-1}(x_2)\big]. \tag{5.10}$$

Again using (5.8), we write

$$\begin{aligned} \frac{\Delta_n^{(2)}(x_1, x_2)}{\Delta_n} &= \frac{\kappa_n \kappa_{n-1}}{h_n h_{n-1}} \left[P_{n-1}(x_1) P_n(x_2) - P_n(x_1) P_{n-1}(x_2)\right] \\ &= \frac{\kappa_n \kappa_{n-1}}{h_n h_{n-1}} \begin{vmatrix} P_{n-1}(x_1) & P_n(x_1) \\ P_{n-1}(x_2) & P_n(x_2) \end{vmatrix}. \end{aligned} \tag{5.11}$$

This formula can be easily extended to the general case of $\Delta^{(n)}(x_1, \ldots, x_n)$; in what follows we shall make use only of formulae (5.9) and (5.11).

Consider now how we can use all these formulae in application to the boundary correlation functions. First we note, following paper [27], that the determinant entering the expression for the homogenous model partition function can be related with orthogonal polynomials using the integral representation

$$\frac{\sin(2\eta)}{\sin(\lambda - \eta)\sin(\lambda + \eta)} = \int_{-\infty}^{\infty} \mathrm{e}^{x(\lambda - \pi/2)} \frac{\sinh(\eta x)}{\sinh(\pi x/2)}\, \mathrm{d}x. \tag{5.12}$$

This formula is valid if $0 < \eta < \pi/2$ and $\eta < \lambda < \pi - \eta$; these values of λ and η correspond to the so-called disordered regime of the six-vertex model (for similar formulae

valid for other regimes, see [28]). This regime is the most interesting especially for combinatorial applications of the six-vertex model with DWBC (in these applications one further specializes to $\lambda = \pi/2$). It can be easily seen that our results below, however, do not depend on the particular choice of the regime, and can be extended to other regimes simply using the proper analytical continuation in the parameters λ and η.

Formula (5.12) implies that we have to deal with the set of polynomials which are orthogonal with respect to the following weight function

$$\mu(x) = \mu(x;\lambda,\eta) = \mathrm{e}^{x(\lambda-\pi/2)}\frac{\sinh(\eta x)}{\sinh(\pi x/2)}. \tag{5.13}$$

The corresponding polynomials $P_n(x) = P_n(x;\lambda,\eta)$ also depend on λ and η which are to be considered as parameters. For later use let us mention the following useful property of these polynomials

$$P_n(x;\lambda,\eta) = (-1)^n\, P_n(-x;\pi-\lambda,\eta). \tag{5.14}$$

This property can be easily established in virtue of formula (5.7). It is to mentioned also that both the leading coefficient $\kappa_n = \kappa_n(\lambda,\eta)$ and the normalization constant $h_n = h_n(\lambda,\eta)$ are invariant under the substitution $\lambda \to \pi - \lambda$.

The transformation $\lambda \to \pi - \lambda$ is related to the so-called crossing symmetry of the six-vertex model which has useful consequences for the one-point boundary correlation function $H_N^{(r)}$. Recall that the the crossing symmetry is the symmetry of the vertex weights under reflection with respect to the vertical axis, and simultaneous interchange of the functions a and b, which is equivalent to setting $\lambda \to \pi - \lambda$. As we shall explain now, these simple properties related to the crossing symmetry allow one to derive easily two equivalent representations for the one-point boundary correlation function $H_N^{(r)}$. Using these formulae we shall then show that the two-point boundary correlation function $H_N^{(r_1,r_2)}$ is expressible in terms of one-point boundary correlators.

Indeed, since the lattice with DWBC is invariant under the reflection with respect to the vertical axis, the crossing symmetry thus imply that the following relation holds

$$H_N^{(r)}(\lambda,\eta) = H_N^{(N-r+1)}(\pi-\lambda,\eta). \tag{5.15}$$

Consider expression (3.21) for the one-point function $H_N^{(r)}$. Due to (5.9) we can rewrite it as

$$H_N^{(r)}(\lambda,\eta) = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r\big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\kappa_{N-1}(\lambda,\eta)}{h_{N-1}(\lambda,\eta)} \times P_{N-1}(\partial_\varepsilon;\lambda,\eta)\frac{(\sin\varepsilon)^{N-r}[\sin(\varepsilon-2\eta)]^{r-1}}{[\sin(\varepsilon+\lambda-\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \quad (5.16)$$

Taking into account (5.14) and the properties of the leading coefficient $\kappa_n(\lambda,\eta)$ and the normalization constant $h_n(\lambda,\eta)$ mentioned above, it can be easily seen that from (5.15) and (5.16) the following expression is valid as well

$$H_N^{(r)}(\lambda,\eta) = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r\big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\kappa_{N-1}(\lambda,\eta)}{h_{N-1}(\lambda,\eta)} \times P_{N-1}(\partial_\varepsilon;\lambda,\eta)\frac{(\sin\varepsilon)^{r-1}[\sin(\varepsilon+2\eta)]^{N-r}}{[\sin(\varepsilon+\lambda+\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \quad (5.17)$$

Note that this expression means simply that the limit $\varepsilon \to 0$ in (5.16) can be changed into $\varepsilon \to 2\eta$ without altering the result.

Thus, these two equivalent representations, (5.16) and (5.17), can be used in the study of the two-point correlation function $H_N^{(r_1,r_2)}$ given by expression (4.23), which certainly involves similar structures.

Before turning to this analysis, let us put the above formulae for the one-point function in a more compact and convenient notations. In what follows we shall often omit the dependence on λ and η where possible.

We define the functions

$$\omega(\epsilon) = \frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)}\,\frac{\sin\varepsilon}{\sin(\varepsilon-2\eta)}, \qquad \varrho(\epsilon) = \frac{\sin(\lambda-\eta)}{\sin(2\eta)}\,\frac{\sin(\varepsilon-2\eta)}{\sin(\varepsilon+\lambda-\eta)}; \quad (5.18)$$

which are related to each other as

$$\varrho(\varepsilon) = \frac{1}{\omega(\varepsilon)-1}. \quad (5.19)$$

Also we define

$$\tilde\omega(\epsilon) = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}\,\frac{\sin\varepsilon}{\sin(\varepsilon+2\eta)}, \qquad \tilde\varrho(\epsilon) = \frac{\sin(\lambda+\eta)}{\sin(2\eta)}\,\frac{\sin(\varepsilon+2\eta)}{\sin(\varepsilon+\lambda+\eta)}; \quad (5.20)$$

which are in turn related to each other as

$$\tilde{\varrho}(\varepsilon) = \frac{1}{1-\tilde{\omega}(\varepsilon)}. \tag{5.21}$$

Note, that the functions with tildes are introduced such that

$$\tilde{\omega}(\varepsilon;\lambda,\eta) = \omega(-\varepsilon;\pi-\lambda,\eta), \qquad \tilde{\varrho}(\varepsilon;\lambda,\eta) = -\varrho(-\varepsilon;\pi-\lambda,\eta) \tag{5.22}$$

in accordance with the crossing symmetry considerations made above. Additionally, let us denote

$$K_{N-1}(x) = (N-1)!\,\varphi^N\,\frac{\kappa_{N-1}}{h_{N-1}}\,P_{N-1}(x) \tag{5.23}$$

where $\varphi = \varphi(\lambda,\eta)$ is exactly the function defining entries of the matrix Φ, see (2.22). In these notations formulae (5.16) and (5.17) for the correlation function $H_N^{(r)}$ read

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon)\big[\omega(\varepsilon)\big]^{N-r}[\varrho(\varepsilon)]^{N-1}\Big|_{\varepsilon=0} \tag{5.24}$$

and

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon)\big[\tilde{\omega}(\varepsilon)\big]^{r-1}[\tilde{\varrho}(\varepsilon)]^{N-1}\Big|_{\varepsilon=0}, \tag{5.25}$$

respectively.

Consider now our main object, the two-point correlation function $H_N^{(r_1,r_2)}$, which is given by formula (4.23). Obviously, function (4.22) contains all the structures introduced above apart from the factor $\sin(\varepsilon_2-\varepsilon_1+2\eta)$ standing in the denominator there. However, using the identity

$$\sin(2\eta)\sin(\varepsilon_2-\varepsilon_1+2\eta) = \sin\varepsilon_1\sin\varepsilon_2 - \sin(\varepsilon_1-2\eta)\sin(\varepsilon_2+2\eta) \tag{5.26}$$

it can be easily seen that

$$\frac{\sin(\varepsilon_1+\lambda-\eta)\sin(\varepsilon_2+\lambda+\eta)}{\sin(\varepsilon_2-\varepsilon_1+2\eta)} = \frac{1}{\varphi\varrho(\varepsilon_1)\tilde{\varrho}(\varepsilon_2)}\,\frac{1}{\omega(\varepsilon_1)\tilde{\omega}(\varepsilon_2)-1}. \tag{5.27}$$

Thus, taking into account formula (5.11) we can write the two-point correlation function in the form

$$\begin{aligned} H_N^{(r_1,r_2)} = {} & [K_{N-1}(\partial_{\varepsilon_1})K_{N-2}(\partial_{\varepsilon_2}) - K_{N-2}(\partial_{\varepsilon_1})K_{N-1}(\partial_{\varepsilon_2})] \\ & \times \frac{\big[\omega(\varepsilon_1)\big]^{N-r_1}[\varrho(\varepsilon_1)]^{N-2}\big[\tilde{\omega}(\varepsilon_2)\big]^{N-r_2}[\tilde{\varrho}(\varepsilon_2)]^{N-2}}{\omega(\varepsilon_1)\tilde{\omega}(\varepsilon_2)-1}\Bigg|_{\varepsilon_1=0,\varepsilon_2=0}. \end{aligned} \tag{5.28}$$

Now taking into account that $\omega(\varepsilon), \tilde{\omega}(\varepsilon) \to 0$ as $\varepsilon \to 0$ we can expand the denominator in (5.28) in power series and it can be easily seen that only the first few terms (actually not more than N) of this expansion will contribute. As a result, in virtue of relations (5.19) and (5.21), we arrive to the following expression in terms of the one-point functions

$$H_N^{(r_1,r_2)} = \sum_{j=1}^{N} \Big(H_N^{(r_1-j+1)} H_{N-1}^{(N-r_2+j)} - H_N^{(r_1-j)} H_{N-1}^{(N-r_2+j)} - H_{N-1}^{(r_1-j)} H_N^{(N-r_2+j+1)} + H_{N-1}^{(r_1-j)} H_N^{(N-r_2+j)} \Big) \quad (5.29)$$

where it is assumed that if $r \leq 0$ or $r \geq N+1$ then $H_N^{(r)} = 0$ by definition. The formula (5.29) is our main result here.

As a comment to this result let us rewrite it in terms of the generating functions. Let us introduce the generating function for the two-point correlation function

$$H_N(u,v) := \sum_{r=1,s=1}^{N} H_N^{(N-r+1,s)} u^{r-1} v^{s-1} \quad (5.30)$$

and define the generating function for the one-point correlation function

$$H_N(u) := \sum_{r=1}^{N} H_N^{(N-r+1)} u^{r-1}. \quad (5.31)$$

Then (5.29) implies that

$$H_N(u,v) = \frac{(u-1)H_N(u) \cdot vH_{N-1}(v) - uH_{N-1}(u) \cdot (v-1)H_N(v)}{u-v} = \frac{1}{u-v} \begin{vmatrix} (u-1)H_N(u) & uH_{N-1}(u) \\ (v-1)H_N(v) & vH_{N-1}(v) \end{vmatrix}. \quad (5.32)$$

This formula generalize to arbitrary values of the vertex weights the result of paper [25] where an equivalent expression was derived in the case $\lambda = \pi/2$ and $\eta = \pi/6$, i.e., when $a = b = c$ (the so-called ice point).

As a final comment we would like to stress that formula (5.29) implies that doubly refined weighted enumerations of ASMs can be easily obtained from the corresponding singly refined ones. In particular, the explicit expressions for the refined x-enumerations, known for $x = 1, 2, 3$, can be just plugged into this formula to obtain the corresponding doubly refined ones. A discussion of the related one-point boundary correlation functions in application to the refined 1-, 2-, and 3-enumerations of ASMs can be found in [27].

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